

# The Quaternionic Exponential (and beyond) ERRATA & ADDENDA

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## Errata

- Page 2:

The multiplication is defined in such a way that  $e=(1,0,0,0)$  is its neutral element.

- Page 3:

We ask for a norm of a Banach algebra to verify  $|p * q| \leq |p||q|$  and not just that there exists some positive  $k$  such that  $|p * q| \leq k|p||q|$ . As noted the case for real numbers, complex numbers, quaternions and octonions is even better yet, as we have  $|p * q| = |p||q|$ .

- Page 3:

$$\operatorname{Re}(q) = \frac{1}{2}(q + \bar{q}) \text{ and } \operatorname{Ur}(q) = \frac{1}{2}(q - \bar{q}).$$

- Page 6:

The formula is not in error, but might be more readable in the form  $r(\vec{a}) = \cos(\theta)[\vec{a} - (\vec{u} \cdot \vec{a})\vec{u}] + \sin(\theta)(\vec{u} \wedge \vec{a}) + (\vec{u} \cdot \vec{a})\vec{u}$ .

- Page 7:

Instead of  $q = \pm \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ x \\ \sin\left(\frac{\theta}{2}\right) \\ y \\ \sin\left(\frac{\theta}{2}\right) \\ z \\ \sin\left(\frac{\theta}{2}\right) \end{bmatrix}$ , read  $q = \pm \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)x \\ \sin\left(\frac{\theta}{2}\right)y \\ \sin\left(\frac{\theta}{2}\right)z \end{bmatrix}$ .

- Page 19, footnote:

$$\text{Instead of } \text{sinc}_a : \left[ \mathbf{R} \rightarrow \mathbf{R}, x \mapsto \frac{\sin\left(\frac{\pi x}{a}\right)}{\frac{x}{a}} \right], \text{ read } \text{sinc}_a : \left[ \mathbf{R} \rightarrow \mathbf{R}, x \mapsto \frac{\sin\left(\frac{\pi x}{a}\right)}{\frac{\pi x}{a}} \right].$$

- Page 19, footnote:

$$\text{Instead of } \text{sinhc}_a : \left[ \mathbf{R} \rightarrow \mathbf{R}, x \mapsto \frac{\sinh\left(\frac{\pi x}{a}\right)}{\frac{x}{a}} \right], \text{ read } \text{sinhc}_a : \left[ \mathbf{R} \rightarrow \mathbf{R}, x \mapsto \frac{\sinh\left(\frac{\pi x}{a}\right)}{\frac{\pi x}{a}} \right].$$

## Addenda

- More structure

It is interesting to note that  $\mathbf{C}$ ,  $\mathbf{H}$  and  $\mathbf{O}$  are (left) vector spaces over  $\mathbf{C}$ .

The basis of  $\mathbf{H}$  as a left  $\mathbf{C}$ -vector space is  $(1, j)$ , and the basis of  $\mathbf{O}$  as a left  $\mathbf{C}$ -vector space is  $(1, j, e', j')$ . However, if  $q = \alpha + \beta i + \gamma j + \delta k \in \mathbf{H}$ , then  $q = (\alpha + \beta i) + (\gamma + \delta i)j$ , but if  $o = \alpha + \beta i + \gamma j + \delta k + \varepsilon e' + \zeta i' + \eta j' + \theta k' \in \mathbf{O}$  then  $o = (\alpha + \beta i) + (\gamma + \delta i)j + (\varepsilon + \zeta i)e' + (\eta - \theta i)j'$  (note the minus sign in the last factor).

If we write  $q = \Gamma + \Delta j$ , with  $\Delta \in \mathbf{C}$  and  $\Gamma \in \mathbf{C}$ , then  $\bar{q} = \bar{\Gamma} - \Delta j$ , and if we also have  $p = A + B j$ , with  $A \in \mathbf{C}$  and  $B \in \mathbf{C}$ , then  $pq = (A\Gamma - B\bar{\Delta}) + (A\Delta + B\bar{\Gamma})j$ . In particular, if  $z \in \mathbf{C}$  then  $jz = \bar{z}j$ .

Things break down when we want to consider  $\mathbf{O}$  as a structure over  $\mathbf{H}$ , however. Indeed, there is no widely-accepted generalization of vector field where the role of the scalars is taken by a non-commutative structure, as is the case with  $\mathbf{H}$  as most interesting properties of vector spaces fail to remain true in that case, in general (though by requiring the scalars to be merely a commutative ring instead of a full blown field, quite a few properties remain true; this structure is known as a module).

However, if  $o = \alpha + \beta i + \gamma j + \delta k + \varepsilon e' + \zeta i' + \eta j' + \theta k' \in \mathbf{O}$ , then it is also true that  $o = (\alpha + \beta i + \gamma j + \delta k) + (\varepsilon + \zeta i + \eta j + \theta k)e'$ .

- More Geometry

Another interesting way to see  $\mathbf{H}$  is as  $\mathbf{R} \times \mathbf{R}^3$ . In this case, if  $q_1 = (t_1, V_1) \in \mathbf{R} \times \mathbf{R}^3$  and  $q_2 = (t_2, V_2) \in \mathbf{R} \times \mathbf{R}^3$ , then the quaternionic product can be expressed as  $q_1 q_2 = (t_1 t_2 - V_1 \cdot V_2, t_1 V_2 + t_2 V_1 + V_1 \wedge V_2)$ , with “ $\cdot$ ” the scalar product on  $\mathbf{R}^3$  and “ $\wedge$ ” the vector product in  $\mathbf{R}^3$ .

- Finding the quaternions for a given rotation of  $\mathbf{R}^3$

If we are given the rotation in term of vector and angle in  $[0; +\pi]$ , then this has been solved in the main text (if the angle is in  $[-\pi; 0]$ , we just take the opposite of both the angle and vector; the identity and its opposite are trivial to solve). The opposite quaternion is also a solution, of course.

If we are simply given a rotation matrix, then, essentially, we first find its elements (vector and angle), and use the procedure above. To find an invariant vector, we simply solve the linear system which defines them. For the angle, we first find its cosine using the trace. Then we build a vector orthogonal to the invariant vector we found (always possible starting from one of the canonical basis vector and using some classical orthonormalization procedure) to check the sign of the angle.

If we are given a succession of rotation, it may be advantageous in applications to chose among the successions of pairs of opposite solutions that for which the distance between successive quaternions is the smallest.

- More rotations

We have considered  $[\mathbf{H} \rightarrow \mathbf{H}, p \mapsto pq]$  and  $d_q = [\mathbf{H} \rightarrow \mathbf{H}, p \mapsto qp]$  in the first chapter and seen, thru some amount of computation, that they gave rise to rotations on  $\mathbf{R}^4$ , when  $|q|=1$ .

We present here another take on the same subject, aimed at giving effective methods of parameterizing  $\mathbf{SO}(4, \mathbf{R})$

It is also interesting to consider  $g_q = [\mathbf{H} \rightarrow \mathbf{H}, p \mapsto p\bar{q}]$ , as both  $g_q$  and  $d_q$  are  $\mathbf{C}$ -linear operators on  $\mathbf{H}$ , which trivially verify  $g_{qq'} = g_q \circ g_{q'}$  and  $d_{qq'} = d_q \circ d_{q'}$ . We can also very simply verify that  $(g_q(p)|g_q(p')) = \|q\|^2 (p|p') = (d_q(p)|d_q(p'))$ . Obviously  $g_1$  and  $d_1$  are both the identity on  $\mathbf{H}$ . Considering them now as  $\mathbf{R}$ -linear operators on  $\mathbf{H}$ , we see that their determinant in the canonical basis must stay of the same sign on  $\mathbf{S}^3$ , hence must stay positive (since  $1 \in \mathbf{S}^3$ ), therefore must be always equal to 1 on  $\mathbf{S}^3$ .

Hence  $[\mathbf{S}^3 \rightarrow \mathbf{SO}(4, \mathbf{R}), q \mapsto g_q]$  and  $[\mathbf{S}^3 \rightarrow \mathbf{SO}(4, \mathbf{R}), q \mapsto d_q]$  are both group homeomorphism, sending 1 to the identity.

Using the same kind of topological argument as in the case of  $\mathbf{SO}(3, \mathbf{R})$ , we get the parameterization of  $\mathbf{SO}(4, \mathbf{R})$  that we announced ([M. Berger (1990)], [C. Godbillon (1971)]) simply by considering  $(p, q) \mapsto d_p \circ g_q$ , save for the determination of the kernel. We will, however, aim here for a more constructive approach to surjectivity.

Given  $r$  a rotation of  $\mathbf{R}^4$ , we seek  $p \in \mathbf{S}^3$  and  $q \in \mathbf{S}^3$  such that for any quaternion  $s$  we have  $ps\bar{q} = r(s)$ . Hence, by applying that to  $s=1$ , we find that necessarily  $p = r(1)q$  (since  $\bar{q} = q^{-1}$  as  $|q|=1$ ). Therefore we are led to solve  $qs\bar{q} = \rho(s)$  for all  $s$ , with  $\rho(s) = \overline{r(1)}r(s)$  (as  $|r(t)| = |t|$  for all quaternion  $t$ , since  $r$  is a rotation, and hence  $|r(1)| = 1$ ). But then  $\rho(\mu) = \overline{r(1)}r(\mu) = \mu \overline{r(1)}r(1) = \mu$  for all  $\mu \in \mathbf{R}$ , which means  $\mathbf{R}$  is invariant, and we know that  $\rho$  is a rotation  $\mathbf{R}^4$ , as the composition of  $r$ , which is one by hypothesis, and the multiplication on the left by a unit quaternion, which is one also as we have seen in the main text. This means we are simply back to solving on  $\mathbf{S}^3$  the equation  $\rho_q = \mathcal{M}(\rho, C, C)$  (with  $q \mapsto \rho_q$  as presented in the first chapter).

Finally, given two unit quaternions  $p = \alpha + \beta i + \gamma j + \delta k$  and  $q = \varepsilon + \zeta i + \eta j + \theta k$ , the rotation matrix on  $\mathbf{R}^4$  is given explicitly by:

$$\begin{bmatrix} \alpha\varepsilon + \beta\zeta + \gamma\eta + \delta\theta & +\alpha\zeta - \beta\varepsilon - \gamma\theta + \delta\eta & +\alpha\eta + \beta\theta - \gamma\varepsilon - \delta\zeta & +\alpha\theta - \beta\eta + \gamma\zeta - \delta\varepsilon \\ -\alpha\zeta + \beta\varepsilon - \gamma\theta + \delta\eta & \alpha\varepsilon + \beta\zeta - \gamma\eta - \delta\theta & -\alpha\theta + \beta\eta + \gamma\zeta - \delta\varepsilon & +\alpha\eta + \beta\theta + \gamma\varepsilon + \delta\zeta \\ -\alpha\eta + \beta\theta + \gamma\varepsilon - \delta\zeta & +\alpha\theta + \beta\eta + \gamma\zeta + \delta\varepsilon & \alpha\varepsilon - \beta\zeta + \gamma\eta - \delta\theta & -\alpha\zeta - \beta\varepsilon + \gamma\theta + \delta\eta \\ -\alpha\theta - \beta\eta + \gamma\zeta + \delta\varepsilon & -\alpha\eta + \beta\theta - \gamma\varepsilon + \delta\zeta & +\alpha\zeta + \beta\varepsilon + \gamma\theta + \delta\eta & \alpha\varepsilon - \beta\zeta - \gamma\eta + \delta\theta \end{bmatrix}$$

### Additional Bibliography

C. Godbillon (1971): Éléments de Topologie Algébrique; Collection Méthodes, Hermann, 1971.